RADIATIVE – CONDUCTIVE HEAT TRANSMISSION IN THE REGULAR MODE OF THE SECOND KIND

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Radiative–conductive heat transmission has been studied theoretically in a plane layer with a linearly varying boundary temperature. It is shown when the regular mode of the second kind prevails. An exact solution is obtained to the quasisteady-state equation for this stage. The error in the approximate description of the temperature is evaluated.

As is well known, in a medium with a linearly varying boundary temperature there is eventually established a regular mode of the second kind, characterized by a constant rate of temperature change at all points in the medium [1]. The theory of this mode is based on the solution of the Fourier equation and, therefore, its applicability is limited to substances in which energy is transmitted by conduction only. The application of this theory to semitranslucent materials in which the radiative mechanism of heat transmission operates along the conductive mechanism is not justified, since here the temperature field is described by a more general equation of radiative–conductive heat transmission [2]. At the same time, methods of analyzing regular modes in thermophysics have been sufficiently well developed, they offer certain unquestionable advantages and their application to semitranslucent materials is dictated by practical considerations. In this article we will analyze theoretically the regularization of the transient radiative–conductive heat transmission in a plane layer the temperature of whose boundary surfaces is a linear function of time. As far as we know, this problem has not been dealt with before.

We consider a plate of semitranslucent material and thickness $2l$ (Fig. 1). In direct thermal contact with its surfaces are opaque but otherwise identical bodies (heaters) which ensure a temperature variation of the $T_S = T_i + b\tau$ kind at the boundaries. The initial temperature distribution in the layer is a given symmetrical function f(x). The reflection coefficient at the boundaries R_V is defined by the Fresnel formulas. The thermophysical properties of the material λ and $c\gamma$ as well as its spectral optical characteristics n_ν and k_ν are assumed known. It has been established earlier that the temperature field is insensitive to temperature variations of n_ν and, therefore, the relation $n_\nu(T)$ will be ignored. The relations $\lambda(T)$, $c(T)$, and $k_\nu(T)$ will be accounted for as follows. Since the temperature difference across the layer is much smaller than the absolute temperature, hence one may consider that $d\lambda/dx = dc/dx = dk_\nu/dx = 0$ and λ, c, k_ν vary with the boundary temperature: $\lambda = \lambda(T_S)$, $c = c(T_S)$, and $k_\nu = k_\nu(T_S)$.

The temperature field in a plane layer is described by the equation of radiative–conductive heat transmission:

$$\operatorname{div}(\bar{q} + \bar{E}) = -c\gamma \frac{\partial T}{\partial \tau}, \tag{1}$$

where, in addition to the thermal flux vector q, we introduce the radiation vector $\bar{E}$, the latter being a nonlinear functional of the temperature distribution T(x). The explicit expression for $\bar{E}$ depends on the nature of the reflection at the boundaries [2]. Without loss in generality, it may be assumed that the reflections are predominantly of the mirror kind. Other possible kinds of reflections are analyzed analogously (see [2]). With a change of variables to $\vartheta(x) = T_S - T(x)$ we obtain

$$\lambda(T_s)\frac{\partial^2 \vartheta}{\partial x^2} + \frac{\partial E}{\partial x} = -\gamma c(T_s)\left(b - \frac{\partial \vartheta}{\partial \tau}\right). \tag{2}$$

Integrating from 0 to x, by virtue of symmetry in the temperature field, we find

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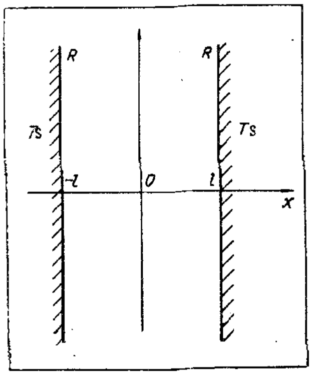

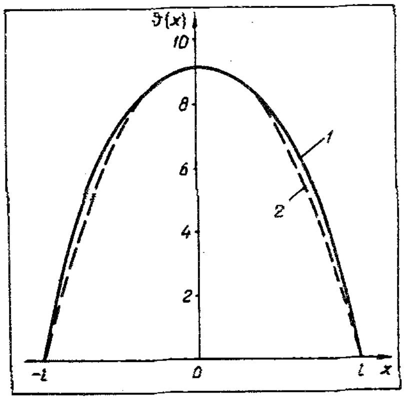

Fig. 1

Fig. 1. Schematic diagram of the problem.

Fig. 2. Temperature distribution during the regular stage (H = 2 cm, b = 800 deg/h, T = 1100°K): 1) exact solution to Eq. (10); 2) parabola $\Delta T(1 - x^2/l^2)$.

$$\lambda \frac{\partial \vartheta}{\partial x} + E(x) - E(0) = -c\gamma b x + c\gamma \frac{\partial}{\partial \tau}\left[\int_0^x \vartheta(\xi, \tau)\, d\xi\right].$$

A subsequent integration from $-l$ to x and the boundary condition $\vartheta(-l) = 0$ yield

$$\lambda\vartheta(x, \tau) + \int_{-l}^{x} [E(\xi, \tau) - E(0, \tau)]\, d\xi = -c\gamma b \frac{(x^2 - l^2)}{2} + c\gamma \frac{\partial}{\partial \tau} \int_{-l} \vartheta(\xi, \tau) K_1(x, \xi)\, d\xi, \tag{3}$$

where function $K_1(x, \xi)$ is defined by the equalities

$$K_1(x, \xi) = \begin{cases} -l - \xi, & -l \leqslant \xi \leqslant 0, \\ x - \xi, & 0 < \xi \leqslant x, \\ 0, & x \leqslant \xi \leqslant l. \end{cases} \tag{4}$$

We will use the expression derived in [3] for the radiation vector. Transformations analogous to those in [4] and the symmetry of the boundary conditions lead to the relation

$$\int_{-l}^{x} [E(\xi, \tau) - E(0, \tau)]\, d\xi = 2\pi \int_{-l}^{l} \int_{\nu=0}^{\infty} n_\nu^2 \{ I_B(\nu, T_s) - I_B[\nu, T(\xi)]\} [K(x, \xi, \nu) - K(-l, \xi, \nu)]\, d\nu d\xi, \tag{5}$$

where

$$K(x, \xi, \nu) = E_3(k_\nu |x - \xi|) + 2 \int_{\varphi=0}^{\pi/2} \frac{R_\nu \sin\varphi \cos\varphi\, e^{-\frac{2k_\nu l}{\cos\varphi}}}{1 - R_\nu^2 e^{-\frac{4k_\nu l}{\cos\varphi}}}$$

$$\times \left\{ \mathrm{ch}\left[\frac{k_\nu (x + \xi)}{\cos\varphi}\right] + R_\nu e^{-\frac{4k_\nu l}{\cos\varphi}} \mathrm{ch}\left[\frac{k_\nu (x - \xi)}{\cos\varphi}\right]\right\} d\varphi,$$

$k_\nu = k_\nu(T_S)$ and $E_3(z)$ is a third-order integroexponential function. Inserting (5) into (3), we obtain a nonlinear integrodifferential equation describing the temperature distribution in the layer in the general formulation of the problem. As has been mentioned earlier, the stipulation that $\vartheta(\xi) \ll T_S$ holds at every point in the plate, so that a linearization of expression (5) cannot significantly influence the calculation of the temperature field. In this case we arrive at the following linear integrodifferential equation:

$$\lambda(T_s)\vartheta(x, \tau) + 2\pi \int_{-l}^{l} \vartheta(\xi, \tau) \left\{ \int_{\nu=0}^{\infty} n_\nu^2 \left[\frac{\partial I_B(\nu, T)}{\partial T}\right]_{T_s}\right.$$

$$\times \left[K(x,\ \xi,\ \nu)-K(-l,\ \xi,\ \nu)\right] d\nu \Big\} d\xi = -\gamma c(T_s)\, b\,\frac{(x^2-l^2)}{2} + \gamma c(T_s)\frac{\partial}{\partial\tau}\int_{-l}^{l} \vartheta(\xi,\ \tau) K_1(x,\ \xi)\, d\xi. \qquad (6)$$

The solution to this equation will be sought in the form $\vartheta(x,\ \tau) = \vartheta_1(\tau)\vartheta_2(x)$. Inasmuch as $\vartheta(x,\ \tau)$ is a symmetrical function for every τ, we will approximate $\vartheta_2(x)$ by a parabola $1 - x^2/l^2$. Inserting $\vartheta_1(\tau)(1 - x^2/l^2)$ into Eq. (6), we see that this equation is automatically satisfied for $x = \pm l$. We require now that the chosen function satisfy this equation also at $x = 0$ (the collocation method). It will be shown subsequently that the error of the thus approximated solution is insignificant. We have

$$\lambda\vartheta_1(\tau) + 2\pi\vartheta_1(\tau)\int_{-l}^{l}\left(1-\frac{\xi^2}{l^2}\right)\Big\{\int_{\nu=0}^{\infty} n_\nu^2\left[\frac{\partial I_B}{\partial T}\right]_{T_s}$$

$$\times \left[K(0,\ \xi,\ \nu)-K(-l,\ \xi,\ \nu)\right] d\nu \Big\} d\xi = \gamma c b\frac{l^2}{2} + \gamma c\frac{\partial\vartheta_1}{\partial\tau}\int_{-l}^{l}\left(1-\frac{\xi^2}{l^2}\right) K_1(0,\ \xi)\, d\xi$$

and, after a few transformations, the following differential equation for determining the function $\vartheta_1(\tau)$:

$$\frac{d\vartheta_1(\tau)}{d\tau} = \Phi(\tau)\,\vartheta_1(\tau) + \frac{6}{5}b, \qquad (7)$$

where

$$\Phi(\tau) = -\frac{12}{5l^2\gamma c(T_s)}\left(\lambda(T_s) + 2\pi\int_{-l}^{l}\left(1-\frac{\xi^2}{l^2}\right)\Big\{\int_{\nu=0}^{\infty} n_\nu^2\left[\frac{\partial I_B}{\partial T}\right]_{T_s}\left[K(0,\ \xi,\ \nu)-K(-l,\ \xi,\ \nu)\right] d\nu\Big\} d\xi\right), \qquad (8)$$

while c, λ, k_ν, and $\partial I_B/\partial T$ depends on the time implicitly through T_s. Solving (7), we find

$$\vartheta_1(\tau) = f(0)\exp\left(\int_0^\tau \Phi(y)\, dy\right) + \frac{6}{5}b\int_0^\tau \exp\left(\int_y^\tau \Phi(\xi)\, d\xi\right) dy. \qquad (9)$$

It is interesting to note that, when $k_\nu \to \infty$, $K(x,\ \xi,\ \nu) \to 0$ for every x. Then (9) yields a usual expression for an opaque medium (discounting the relations $\lambda(T_s)$ and $c(T_s)$):

$$\vartheta_1^*(\tau) = f(0)\, e^{-\frac{12a}{5l^2}\tau} + \frac{bl^2}{2a}\left(1 - e^{-\frac{12a}{5l^2}\tau}\right),$$

which, as τ increases, becomes the well-known temperature distribution in the regular mode of the second kind:

$$\vartheta^*(\tau) = \frac{bl^2}{2a}\left(1-\frac{x^2}{l^2}\right).$$

In order to explain the dynamics of the temperature field when both heat-transmission components are effective, formulas (8) and (9) were calculated on a computer with various values for the system parameters. These calculations have shown that $\vartheta_1(\tau)$ does not, generally, tend towards a constant value, but the derivative $\partial\vartheta_1(\tau)/\partial\tau$ becomes much smaller than b and remains so for a long time period during which the specimen is heated by 500-700°C. Unlike in opaque bodies, where the irregular mode is exponential in character, in semitranslucent media the function $\vartheta_1(\tau)$ is of the extremal kind: the first stage of the process comes to an end after the maximum value has been reached. The heating of the layer from various initial temperature distributions does not exceed 100°C during this stage. The subsequent departure of $\partial\vartheta_1/\partial\tau$ from b does not exceed 2%. For this reason, one may assert within this accuracy that a regular mode prevails in the layer. The values of $\vartheta_1(\tau)$ at various instants of time τ are given in Table 1 for a plate of grade KV quartz glass 2 cm thick (T_i = 500°K, b = 800 deg/h).

Having established that the mode is regular, one can significantly simplify the mathematical description of the heat transmission in this stage. Assuming $\partial\vartheta/\partial\tau = 0$ in Eqs. (3) and (6), one can replace the integrodifferential equations by integral equations.

TABLE 1. Maximum Temperature Difference as a Function of Time (b = 800 deg/h, H = 1 cm)

sec	0	60	120	180	240	300	360	420	480	540
°C	0	13,1	17,65	18,83	19,66	19,81	19,85	19,83	19,80	19,78
sec	600	660	720	1620	1680	1740	2160	2220	2280	
°C	19,74	19,69	19,64	18,65	18,59	18,52	18,07	18,00	17,93	

Let us further analyze the linearized equation of radiative–conductive heat transmission in the regular mode. With the aid of (6) we find

$$\vartheta(x)=\frac{c\gamma b}{2\lambda}(l^2-x^2)-\frac{2\pi}{\lambda}\int\limits_{-l}^{l}\vartheta(\xi)\left\{\int\limits_{\nu=0}^{\infty}n_\nu^2\left(\frac{\partial I_B}{\partial T}\right)_{T_S}[K(x,\ \xi,\ \nu)-K(-l,\ \xi,\ \nu)]\,d\nu\right\}d\xi. \tag{10}$$

In our case $T_S(\tau)$ is a parameter of the equation which determines the values of $\partial I_B/\partial T$, λ, c, and k. In other words, the temperature distribution in the layer based on (10) refers to a definite instant of time and, consequently, to a specific value of $T_S(\tau)$ which affects $\vartheta(x)$ implicitly through λ, c, k_ν, and $\partial I_B/\partial T$. For this reason, Eq. (10) may be considered a quasisteady-state equation. It has been solved on a computer by the method of quadratures. An application of the Simpson and of the Markov rule has shown the latter to be preferable. As the number of nodes N was increased, the resultant temperature distribution stabilized so that, after N = 16, a further increase of N had almost no effect on $\vartheta(x)$.

Curve 1 in Fig. 2 represents the exact solution to Eq. (10). In order to simplify the calculation, we used the "gray" approximation with function k replaced by the mean spectral value k. Such a substitution was inconsequential here, since the properties of the equation and the method of solution were of chief interest. In order to obtain specific results for various semitranslucent materials, however, it would be necessary to consider the selectivity of optical characteristics – as has been shown earlier [3].

Curve 2 in Fig. 2 is the approximating parabola $\Delta T(1-x^2/l^2)$ where $\Delta T = T_S - T(0)$. As can be seen, the departures of $\vartheta(x)$ from the parabola are insignificant. The same result was obtained in an analysis of $\vartheta(x)$ at various instants of time. Thus, it has been established that the solution of the transient equation (6) by the collocation method, as shown here, contains a small error.

If this approximation is used in Eq. (10) and, besides, x = 0 is assumed, then one arrives at the following relation:

$$\Delta T=\frac{bl^2}{2a}-\frac{2\pi}{ac\gamma}\,\Delta T\int\limits_{\nu=0}^{\infty}n_\nu^2\left(\frac{\partial I_B}{\partial T}\right)_{T_S}\left\{\int\limits_{-l}^{l}\left(1-\frac{\xi^2}{l^2}\right)[K(x,\ \xi,\ \nu)-K(-l,\ \xi,\ \nu)]\,d\xi\right\}d\nu, \tag{11}$$

from which follows the equality

$$a=\frac{bl^2}{2\Delta T}-\frac{2\pi}{c\gamma}I(R,\ T,\ k_\nu), \tag{12}$$

where function $I(R, T, k_\nu)$ denotes the integral factor in (11). Relation (12) can be used for determining the thermal diffusivity of semitranslucent materials on the basis of the regular mode of the second kind.

NOTATION

λ is the thermal conductivity;
a is the thermal diffusivity;
c is the specific heat;
γ is the density of the material;
n is the refractive index;
k is the absorption coefficient;
R is the reflection coefficient at the boundary;
$\bar{q}$ is the thermal flux vector;
$\bar{E}$ is the radiation flux vector;

$I_B(\nu, T)$ is the Planck function;
b is the rate of temperature change at the surface;
$2l$ is the layer thickness;
$E_3(z)$ is the third-order integroexponential function;
N is the number of nodes in the quadrature formula;
x, ξ are the space coordinates;
τ is the time coordinate.

Subscripts

p refers to the surface temperature;
i refers to the initial temperature;
ν refers to spectral quantities.

Superscript

* refers to an opaque medium.

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